Particle Swarm Optimization with Inertia Weight and Constriction Factor

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Abstract

In the original Particle Swarm Optimization (PSO) formulation, convergence of a particle towards its attractors is not guaranteed. A velocity constraint is successful in controlling the explosion, but not in improving the fine-grain search. Clerc and Kennedy studied this system, and proposed constriction methodologies to ensure convergence and to fine tune the search. Thus, they developed different constriction methods according to the correlations among some coefficients incorporated to the system. *Type 1"* constriction became very popular because the basic update equations remained virtually unmodified, and the original and intuitive metaphor valid. The main drawbacks of this constriction type are that constriction becomes too strong very quickly as the acceleration is increased; that the *speed of convergence* cannot be easily controlled; and that there is no flexibility to set a desired *form of convergence*. Another problem is that the method can be found in the literature formulated so as to constrict a PSO system which already includes the inertia weight, for which the calculation of a constriction factor using the formulae provided by Clerc and Kennedy does not guarantee convergence. This paper analyzes *Type 1"* constriction in detail, for which *Type 1* constriction is also relevant. The formulae for *Type 1* and *Type 1"* constriction factors suitable for a PSO algorithm including the inertia weight are provided.

Keywords

Constriction factor, guaranteed convergence, inertia weight, particle swarm.

1 Introduction

Particle Swarm Optimization (PSO) is a Swarm Intelligence (SI)-based method inspired by the cooperative behaviour observed in social animals in nature. The *Original PSO* (*OPSO*) algorithm is unstable, and particles tend to diverge from the attractor (the so-called *explosion*). The incorporation of the velocity constraint prevents the explosion but does not improve the poor fine-grain search. In *Classical PSO* (*CPSO*), the incorporation of the inertia weight allows for coefficients settings that ensure convergence and control the desired balance between explorative and exploitative behaviour (refer to [1–5]). Similarly, the *Constricted Original PSO* (*COPSO*) incorporates a constriction factor to the *OPSO*, which ensures convergence and improves the fine-tuning of the search (refer to [6]). As an intended generalization, formulations of the PSO algorithm with both inertia weight and constriction factor can be found in the literature (e.g. [7–12]), where either no attention is paid to convergence issues or the formula presented to calculate a constriction factor that supposedly would guarantee convergence is plainly wrong. This is because the formula proposed by Clerc and Kennedy [6], which is only valid for the *OPSO* ($w = 1$), is applied to the *Constricted Classical PSO* (*CCPSO*). The latter is a PSO algorithm with inertia weight and constriction factor.

The remainder of this paper is organized as follows: The PSO method is reviewed in section 2, where the *OPSO* is presented in 2.1; the classical formulation (*CPSO*) is presented in section 2.2; and the *COPSO* is discussed in section 2.3, including *Type 1* and *Type 1"* constriction classes; a formal study of the *CCPSO* (i.e. PSO with inertia weight and constriction factor) is offered in section 3; and concluding remarks are provided in section 4.

This study addresses misunderstandings, misformulations, and/or misuses of the constriction factor; helps to understand its meaning as envisioned by Clerc and Kennedy [6] and how it works in conjunction with the inertia weight. Finally, it provides the reader with the appropriate formulae to calculate a constriction/scaling factor (χ) which guarantees convergence for any setting of the inertia weight (w) and of the acceleration coefficient (ϕ) in *Constricted Classical PSO* (*CCPSO*).

2 Particle Swarm Optimization

The PSO method was proposed by Kennedy and Eberhart [13], inspired by bird flock simulations aimed at studying social behaviour. The method was also influenced by other simulations, experiments, and theories in social psychology (refer to [14, 15]). Computational Intelligence (CI) techniques such as Evolutionary Algorithms (EAs) also influenced its development once the method was thought of as a SI-based optimizer. Hence the PSO paradigm also has ties to Artificial Intelligence (AI) and mathematical optimization.

2.1 Original Particle Swarm Optimization

The formulation of the *OPSO* algorithm [13] is as follows:

$$\begin{cases} v_{ij}^{(t)} = v_{ij}^{(t-1)} + iw \cdot U_{(0,1)} \cdot \left(pb_{ij}^{(t-1)} - x_{ij}^{(t-1)}\right) + sw \cdot U_{(0,1)} \cdot \left(gb_{j}^{(t-1)} - x_{ij}^{(t-1)}\right) \\ x_{ij}^{(t)} = x_{ij}^{(t-1)} + v_{ij}^{(t)} \end{cases} \tag{1}$$

where

$v_{ij}^{(t)}$, $x_{ij}^{(t)}$: component/coordinate j of the *velocity* and *position*, respectively, of particle i at time-step t;

$pb_{ij}^{(t)}$, $gb_{j}^{(t)}$: coordinate j of the *best experience of particle* i and of the *best experience in the swarm*, respectively, by time-step t;

$iw = sw = 2$: *individuality weight* and *sociality weight*, respectively, both originally equal to 2 [13];

$U_{(0,1)}$: random number generated form a uniform distribution in the range [0,1], resampled anew every time it is referenced.

This system is unstable, and particles tend to diverge from the attractor. The first strategy used to control this *explosion* consisted of limiting the size of each component of a particle's velocity. This helps prevent the explosion but does not help with the convergence or fine-tuning of the search.

Kennedy [16] performed a detailed analysis of the trajectory of a deterministic particle pulled by stationary attractors on a one-dimensional space. He considered both attractors identical (p), and carried out an observational analysis of the trajectory of a particle with initial position $x^{(0)} = p$. Since there is a single attractor, there is also a single acceleration coefficient (ϕ), which is assumed to be constant. Kennedy [16] noticed that the particle displays (pseudo) cyclic behaviour for $0 < \phi < 4$ and diverges for $\phi = 0$ and $\phi \geq 4$, and carried out a detailed analysis of the emerging patterns.

Ozcan and Mohan [17, 18] presented the first theoretical analysis of the dynamic behaviour of the *OPSO*, studying the same simplified system as Kennedy [16]. They studied the trajectory of the same deterministic particle for the same particular case of $x^{(0)} = p$. Under this assumption, they analyzed the trajectory for a few particular values of ϕ claimed to comprise boundary cases for different particle's behaviours, presenting the particle's trajectory equation and stating that the trajectory under these conditions is a sinusoidal wave where the settings of ϕ determine the amplitude and frequency of the wave. Thus, they argue that the particle does not really fly over the search-space but surfs it on sine waves. The particle is then attracted by the weighted average of the two best experiences, moving in step sizes randomly obtained from a sinusoidal wave. The type of wave caught would be determined by the random weights, while the $v_{\max}$ constraint helps the particle jump onto another wave.

2.2 Classical Particle Swarm Optimization

In *CPSO*, the swarm's best experience is replaced by the best experience in the i^{th} particle's neighbourhood. In addition, the inertia weight (w) initially proposed in [19] is incorporated, which allows controlling the explosion and the convergent behaviour. Given that this is the most widespread formulation, it is referred to here as the *CPSO*. The current trend is to present the single essential update equation rather than the traditional separate ones, as shown in (2):

$$x_{ij}^{(t)} = x_{ij}^{(t-1)} + w_i^{(t)} \cdot \left(x_{ij}^{(t-1)} - x_{ij}^{(t-2)}\right) + iw_i^{(t)} \cdot U_{(0,1)} \cdot \left(pb_{ij}^{(t-1)} - x_{ij}^{(t-1)}\right) + sw_i^{(t)} \cdot U_{(0,1)} \cdot \left(lb_{ij}^{(t-1)} - x_{ij}^{(t-1)}\right) \quad (2)$$

where

$lb_{ij}^{(t)}$: coordinate j of the *best experience* in the neighbourhood of particle i up to time-step t;

$w_i^{(t)}$: *inertia weight* of particle i at time-step t;

$iw_i^{(t)}$, $sw_i^{(t)}$: *individuality weight* and *sociality weight*, respectively, of particle i at time-step t.

To the best of our knowledge, the first theoretical study in *CPSO* was presented by van den Bergh [1], who also based his studies on a deterministic particle. He presented the trajectory equation for linearly independent eigenvalues. He also provided a convergence region in the 'ϕ–w' plane, although truncated (only $w > 0$) and without formal definition of its boundaries. Trelea [2] studied the *CPSO* in terms of a deterministic particle as well, using results from the *Dynamic System Theory*. He presented a convergence triangle, without further details. Innocente and Sienz [5] encompassed and extended the theoretical studies of a deterministic particle. They presented all three trajectory equations for the three types of solutions of the recurrence relations in terms of the two essential initial conditions; they formally derived and fully bound the complex and convergence regions in the 'ϕ–w' plane; they studied the trajectories of the particle on the three boundaries of the convergence region and the speed and form of convergence inside, aided by the trajectory equations and by trajectory plots; and finally proposed a means of setting the coefficients (ϕ and w) so as to obtain a desired behaviour.

2.3 Constricted Original Particle Swarm Optimization

Clerc and Kennedy [6] analyzed the trajectory of a deterministic particle for the *OPSO* algorithm, and developed constriction factors aiming to ensure convergence. They built a system of two recurrence relations of first order that describes the simplified system by using the change of variables in (3). Thus, the system of two linear first order recurrence relations is as shown in (4).

$$y^{(t)} = p - x^{(t)} \quad (3)$$

$$\begin{cases} v^{(t)} = v^{(t-1)} + \phi \cdot y^{(t-1)} \\ y^{(t)} = -v^{(t-1)} + (1-\phi) \cdot y^{(t-1)} \end{cases} \quad (4)$$

The system can be expressed in matrix notation as in (5) and (6), while the eigenvalues of the system matrix M are offered in (7) where $i = 1$ for the positive sign.

$$P^{(t)} = \begin{bmatrix} v^{(t)} \\ y^{(t)} \end{bmatrix} = M \cdot P^{(t-1)} = M^t \cdot P^{(0)} \quad (5)$$

$$M = \begin{pmatrix} 1 & \phi \\ -1 & (1-\phi) \end{pmatrix} \quad (6)$$

$$e_i = 1 - \frac{\phi}{2} \pm \frac{\sqrt{\phi^2 - 4\cdot\phi}}{2} = 1 - \frac{\phi}{2} \pm \frac{\gamma}{2} \quad ; \quad i = 1,2 \tag{7}$$

By diagonalizing *M*, Clerc and Kennedy [6] showed that the position of a particle depends on the initial conditions and on its eigenvalues raised to the power of the time-step. If at least one eigenvalue is not smaller than one, the system does not converge. In these cases, they proposed to build a surrogate system whose eigenvalues (e'_1 and e'_2) are both smaller than one. For this purpose, they added five coefficients to the system, whose values can be chosen so as to ensure convergence. Such a system is shown in (8), while *M* is as in (9). Note that γ in (8) and (9) is different from γ in (7).

$$\begin{cases} v^{(t)} = \alpha\cdot v^{(t-1)} + \beta\cdot\phi\cdot y^{(t-1)} \\ y^{(t)} = -\gamma\cdot v^{(t-1)} + (\delta - \eta\cdot\phi)\cdot y^{(t-1)} \end{cases} \tag{8}$$

$$M = \begin{pmatrix} \alpha & \beta\cdot\phi \\ -\gamma & (\delta - \eta\cdot\phi) \end{pmatrix} \tag{9}$$

Then, if the system in (4) does not comply with the convergence condition of both eigenvalues being smaller than one, *constriction coefficients* are applied as in (10), where the eigenvalues of the surrogate system are forced to have magnitudes smaller than one.

$$e'_i = \chi_i \cdot e_i \tag{10}$$

According to how the added coefficients (α, β, γ, δ, η) are correlated, Clerc and Kennedy [6] studied different constriction types; namely, *Type 1* and its derivations *Type 1'* and *Type 1"*, and *Type 2*. For details on these types, refer to their original work [6]. The only constriction considered here is *Type 1"* because it is the only one that maintains the original, intuitive concept of the velocity as the difference between two consecutive positions. Indirectly, *Type 1* constriction is also relevant here.

2.3.1 Type 1 constriction

The coefficients incorporated are correlated as in (11). Clerc and Kennedy [6] suggest that setting the correlations as in (12) ensures real coefficients. Therefore, the system matrix *M* is as in (13).

$$\begin{cases} \alpha = \delta \\ \beta\cdot\gamma = \eta^2 \end{cases} \tag{11}$$

$$\alpha = \beta = \gamma = \eta = \delta = \chi_1 = \chi_2 = \chi \tag{12}$$

$$M = \begin{pmatrix} \chi & \chi\cdot\phi \\ -\chi & \chi\cdot(1-\phi) \end{pmatrix} \tag{13}$$

This constriction type is particularly interesting because the eigenvalues of *M* in (13) are as in (14). Hence the constriction factor that ensures convergence can be calculated as in (15).

$$e'_i = \chi\cdot\left(1 - \frac{\phi}{2} \pm \frac{\sqrt{\phi^2 - 4\cdot\phi}}{2}\right) = \chi\cdot e_i \tag{14}$$

$$\chi = \frac{\kappa}{\max(\|e_1\|,\|e_2\|)} \quad ; \quad 0 < \kappa < 1 \tag{15}$$

For $\phi \geq 4$, the eigenvalues of the original system are real-valued, and $\|e_2\| > \|e_1\|$. Therefore,

$$\chi = \frac{\kappa}{\|e_2\|} = \frac{\|e'_2\|}{\|e_2\|} = \frac{2\cdot\kappa}{\phi - 2 + \sqrt{\phi^2 - 4\cdot\phi}} \quad ; \quad \begin{cases} 0 < \kappa < 1 \\ \phi \geq 4 \end{cases} \tag{16}$$

For $0 < \phi < 4$, the eigenvalues of the original system are complex conjugates with $\|e_1\| = \|e_2\| = 1$. Hence χ is as in (17):

$$\chi = \frac{\kappa}{\|e_i\|} = \kappa \quad ; \quad \begin{cases} 0 < \kappa < 1 \\ 0 < \phi < 4 \end{cases} \tag{17}$$

To summarize, the *Type 1* constriction factor that ensures convergence is calculated as in (18):

$$\chi = \begin{cases} \dfrac{2\cdot\kappa}{\phi - 2 + \sqrt{\phi^2 - 4\cdot\phi}} & \text{if } \phi \geq 4 \\ \kappa & \text{if } 0 < \phi < 4 \end{cases} \quad ; \quad 0 < \kappa < 1 \tag{18}$$

If the eigenvalues of the original system are complex conjugates or real-valued, so are those of the surrogate system. This constriction factor is simply the ratio between the corresponding eigenvalues of both systems. However, the classical PSO formulation is noticeably disrupted, becoming as in (19):

$$x^{(t)} = p - \chi\cdot\left(p - x^{(t-1)}\right) + \underbrace{\chi\cdot\left[\overbrace{-\left(p - x^{(t-1)}\right) + \chi\cdot\left(p - x^{(t-2)}\right)}^{v^{(t-1)}} + \phi\cdot\left(p - x^{(t-1)}\right)\right]}_{v^{(t)}} \tag{19}$$

2.3.2 Type 1" constriction

For this constriction type, the added coefficients are correlated as in (20). For simplicity, Clerc and Kennedy [6] also suggest setting (21). Therefore, with χ instead of α, the system matrix is as in (22) and the eigenvalues as in (23).

$$\alpha = \beta = \gamma = \eta \tag{20}$$

$$\delta = 1 \tag{21}$$

$$M = \begin{pmatrix} \chi & \chi\cdot\phi \\ -\chi & (1 - \chi\cdot\phi) \end{pmatrix} \tag{22}$$

$$e'_i = \frac{1 + \chi - \chi\cdot\phi \pm \sqrt{(\chi\cdot\phi)^2 - (2 + 2\cdot\chi)\cdot(\chi\cdot\phi) + (\chi - 1)^2}}{2} = \frac{\left(1 + \chi - \chi\cdot\phi \pm \sqrt{\Delta}\right)}{2} \neq \chi\cdot e_i \tag{23}$$

In *Type 1* constriction, if the eigenvalues of the original system are complex conjugates or real-valued, so are those of the constricted one. This is not the case in *Type 1"* constriction. The values of ϕ (depending on χ) for which Δ in (23) equals zero are as in (24):

$$\phi_{\substack{\max \\ \min}} = \left(\frac{1}{\chi} + 1\right) \pm \frac{2}{\sqrt{\chi}} \tag{24}$$

If the eigenvalues of the surrogate system are complex conjugates, their magnitudes are given by the square root of χ. Hence convergence is ensured simply by $\chi < 1$. Therefore enforcing both conditions ensures convergence. Clerc and Kennedy [6] proposed using the same constriction factor as for the *Type 1* constriction (see (18)).

If $\phi < 4$, $\Delta < 0$ for $\phi_{\min} < \phi < \phi_{\max}$. In this case, solving (24) is easy, since $\chi = \kappa$. It is also easy to see from (24) that $\phi_{\min} < 4$ for $\phi < 4$ and $1/9 < \kappa < 1$, as stated in [6] and showed in (25):

$$\text{if } \left(\phi < 4 \ \wedge \ \frac{1}{9} < \kappa < 1\right), \ \begin{cases} 0 < \phi_{\min} < 4 \\ \phi_{\max} > 4 \end{cases} \tag{25}$$

If $\phi = 4$, χ still equals κ (see (18)), and therefore there is continuity in the curves 'ϕ–Δ'.

If $\phi > 4$, $\Delta < 0$ for $4 < \phi < \phi_{\min}$, with $\phi_{\min}$ as in (24). However, the calculation of $\phi_{\min}$ is not straightforward for $\phi > 4$ (see (24) and (18)), with $\phi_{\min}$ increasing as κ increases. Clerc and Kennedy [6] computed the values of $\phi_{\min}$ for $\kappa = 0.40$ ($\phi_{\min} = 8.07$) and $\kappa = 0.99$ ($\phi_{\min} = 39799.76$). They referred to these values as $\phi_{\max}$ instead because they comprise the upper bound for $\Delta < 0$. We refer to them as $\phi_{\min}$ because they are calculated with the negative sign in (24).

The desirable feature of this constriction is that the classical formulation of PSO is marginally altered, maintaining the intuitive notion of the velocity as the difference between two consecutive positions. The PSO update equations for this constriction type result as shown in (26). Note that the update equations of the *OPSO*, the *Type 1 COPSO*, and the *Type 1" COPSO* coincide for $\chi = 1$. One undesirable feature is that the calculation of χ is not straightforward, as it no longer represents the ratio between the magnitudes of the corresponding eigenvalues of the original and the surrogate systems.

$$x^{(t)} = x^{(t-1)} + \overbrace{\chi \cdot \left[\underbrace{\left(x^{(t-1)} - x^{(t-2)}\right)}_{v^{(t-1)}} + \phi \cdot \left(p - x^{(t-1)}\right) \right]}^{v^{(t)}} \tag{26}$$

Since ϕ is a random variable, it is common practice to replace ϕ by $\phi_{\max}$ in the full PSO system to be on the safe side. Thus convergence is ensured for all randomly generated ϕ. This implies that every generated $\phi < \phi_{\max}$ will be constricted more strongly than necessary. Generalizing for coefficients that may differ for different particles and change over time, the update equations of *Type 1" COPSO* are:

$$x_{ij}^{(t)} = x_{ij}^{(t-1)} + \chi_i^{(t)} \cdot \left[\left(x_{ij}^{(t-1)} - x_{ij}^{(t-2)}\right) + iw_i^{(t)} \cdot U_{(0,1)} \cdot \left(pb_{ij}^{(t-1)} - x_{ij}^{(t-1)}\right) + sw_i^{(t)} \cdot U_{(0,1)} \cdot \left(lb_{ij}^{(t-1)} - x_{ij}^{(t-1)}\right) \right] \tag{27}$$

$$\chi_i^{(t)} = \begin{cases} \dfrac{2 \cdot \kappa_i^{(t)}}{\phi_{\max i}^{(t)} - 2 + \sqrt{\left(\phi_{\max i}^{(t)}\right)^2 - 4 \cdot \phi_{\max i}^{(t)}}} & \text{if } \phi_{\max i}^{(t)} > 4 \\ \kappa_i^{(t)} & \text{otherwise} \end{cases} \quad \text{with } \kappa_i^{(t)} \in (0,1) \tag{28}$$

The *Type 1" COPSO* can be easily reduced to the *CPSO* using the constriction factor as the constricted inertia weight (w_c) and the constricted acceleration coefficient (ϕ_c) instead of the original one (ϕ), as shown in (29):

$$\begin{cases} w_c = \chi \cdot w = \chi \\ \phi_c = \chi \cdot \phi \end{cases} \tag{29}$$

Hence this constriction basically works as a scaling-down factor for the original coefficients of the system ($w = 1$ and ϕ). Therefore convergence is ensured if the pair of constricted coefficients (ϕ_c, w_c) falls within the convergence triangle defined in [5]. Each line in Figure 1 represents all the pairs (ϕ_c, χ) or (ϕ_c, w_c) for a given value of ϕ in the original system, for all values of $\kappa \in (0,1)$. The top-right end of the lines correspond to $\kappa = 1$. Recall that $\chi = \kappa$ for $\phi \leq 4$.

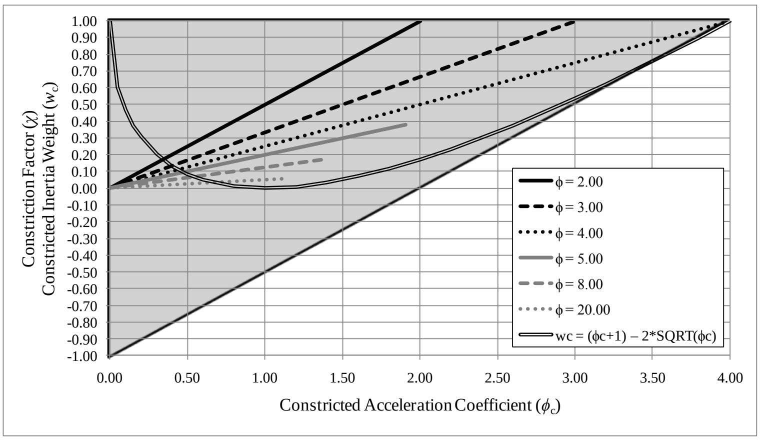

Figure 1: Constriction factor (χ) versus constricted acceleration coefficient (ϕ_c) for different values of the acceleration coefficient (ϕ) in the original system. Each line represents all pairs (ϕ_c, χ) or (ϕ_c, w_c) for all values of $\kappa \in (0,1)$. The top-right end of the lines correspond to $\kappa = 1$. The shaded triangle is the convergence region while the complex conjugate eigenvalues are within the dotted parabola (see [5]).

The convergence region with $w < 0$ as well as some parts of the convergence region with $w \geq 0$ cannot be covered by all possible settings of the *COPSO*. The triangular region enclosed by the top and left boundaries of the convergence region and by the line for $\phi = 4$ in Figure 1 can be covered by scaling the line corresponding to any ($0 < \phi < 4$) with $0 < (\chi = \kappa) < 1$. However, the constriction becomes too strong for $\phi > 4$, and the regions near the right-bottom boundary of the convergence region cannot be covered by *Type 1"* constriction using (18). In fact, not even the whole complex region can be covered. The part of the convergence region that can be covered by the *COPSO* is given by the envelope of the top-right ends of the lines corresponding to all values of ϕ in Figure 1, as shown in Figure 2.

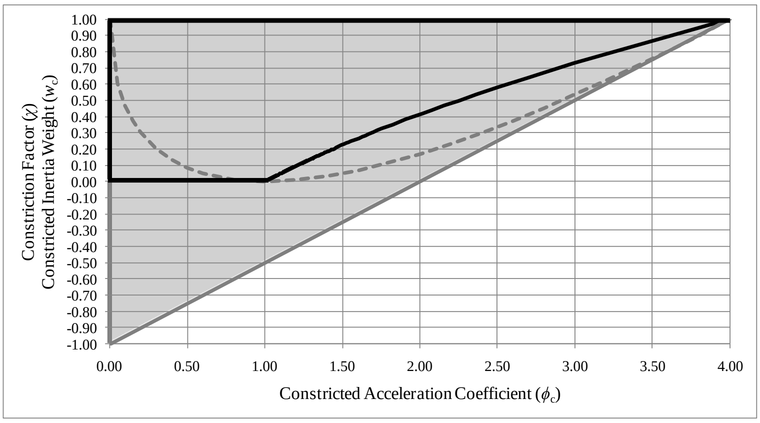

Figure 2: The gray shaded triangle shows the convergence region, whereas the inner black loop encloses the region in the (ϕw) plane which can be covered by Type 1" constriction using (18).

In addition, the impact of different (ϕ,κ) settings over the particle's trajectory is not as straightforward as that of different (ϕ,w) settings within the convergence triangle in *CPSO*, where different behaviours for different sub-regions can be identified (refer to [5]).

3 Constricted Classical Particle Swarm Optimization

It is not infrequent to find in the literature the formulation of the velocity update equation including both inertia weight (w) and constriction factor (χ). However, perhaps due to the fact that following the excellent work presented by Clerc and Kennedy [6] is not straightforward, their constriction factor is often misinterpreted and therefore miscalculated and/or misused. In some cases, the formulation is incorrect but never used whereas in others there is no calculation of the constriction factor but simply some fixed or adaptive setting, disregarding its convergence properties.

Zhang et al. [8] formulate the *CCPSO* with a *Type 1"* constriction factor as in (28) (with $\kappa = 1$), which is only valid for $w = 1$. In *CCPSO*, the range of values of ϕ for which the eigenvalues are complex conjugates (see (24)) does not only depend on χ but also on w. In addition, the eigenvalues of the surrogate system being complex conjugates with $\chi < 1$ no longer guarantees that their magnitude be smaller than one because they are no longer given by $\sqrt{\chi}$ but by $\sqrt{\chi \cdot w}$. Therefore the calculation of χ as in *COPSO* does not apply to *CCPSO*. However, later in their experiments, they simply use $\chi = 1$ and random inertia weights. Chen et al. [9 p. 308] also formulate the *CCPSO* with a *Type 1"* constriction factor as in (28), with $\kappa = 1$. Beware that they called the constriction factor κ and ignored the coefficient κ in (28) altogether. In their experiments, they do not use that formulation and simply implement an adaptive constriction factor within experimentally obtained limits [0.1,1.0], with linearly decreasing w in the range [0.9,0.4] and $\phi \in [0.0,4.1]$, disregarding convergence issues. Mahor et al. [10 p. 2137] also posed the *CCPSO* with a *Type 1"* constriction factor as in (28) (with $\kappa = 1$) with linearly decreasing w. Higashi and Iba [7 p.72] set $w = 0.9$, $\phi_{max} = 4$, and χ as random numbers from

0.9 to 1.0. Bui et al. [12 p. 336] used a *CCPSO* with an adaptive χ with deterministic decay rules (w = 0.729, $\phi_{\max}$ = 4.1). Beielstein et al. [11] formulate the *CCPSO* with a *Type 1"* constriction factor, but use χ = 1 in their experiments.

The formulae provided hereafter ensure convergence of a deterministic particle for any value of ϕ. For the full PSO system, it suffices to replace ϕ by $\phi_{\max}$ to ensure convergence.

3.1 Type 1 CCPSO

This constriction factor is such that (30) is satisfied and e'_i < 1. However, the eigenvalues of the original system (e_i) are no longer calculated as in (7) but as in (31). Therefore, the calculation of this constriction type remains as in (15), simply modifying the calculation of the eigenvalues to account for the inertia weight (w). Recall that for this constriction type, the PSO update equation is as in (19).

$$\chi = \frac{e'_i}{e_i} = \frac{\kappa \in (0,1)}{e_i} \tag{30}$$

$$e_i = \frac{(1+w-\phi) \pm \sqrt{\phi^2 - (2\cdot w + 2)\cdot \phi + (w-1)^2}}{2} \quad ; \quad i = 1,2 \tag{31}$$

3.2 Type 1" CCPSO

Constricting the *CPSO* with *Type 1"* constriction is not as straightforward, since (30) is not valid. Although Clerc and Kennedy [6] used the same constriction as for *Type 1* constriction, this only ensures convergence for the *OPSO*. *Type 1"* constriction may be viewed as scaling w = 1 and ϕ by a factor of χ. Let us call w_0 and ϕ_0 the values of w and ϕ initially set (i.e. of the original system). Their constricted values w_c and ϕ_c would be on the line defined as in (32):

$$w = \frac{w_0}{\phi_0} \cdot \phi \tag{32}$$

The pairs (ϕ,w) on this line which ensure convergence are within the segment defined between the origin (0,0) and the intersection of (32) with either the top or the right-bottom lines bounding the convergence region in Figure 1. Thus, the *uppermost* point (A_2) defining the segment of values (ϕ_c,w_c) which ensure convergence is given by (33), whereas the *lowermost* point is A_1 = (0,0).

$$A_2 = \left(\phi_{\max}^{(conv)}, w_{\max}^{(conv)}\right) = \begin{cases} \left(\left[\dfrac{\phi_0}{w_0}\right], [1]\right) & \text{if } \dfrac{w_0}{\phi_0} > \dfrac{1}{4} \\ \left(\left[\dfrac{2\cdot\phi_0}{\phi_0 - 2\cdot w_0}\right], \left[\dfrac{2\cdot w_0}{\phi_0 - 2\cdot w_0}\right]\right) & \text{otherwise} \end{cases} \tag{33}$$

The pairs (ϕ,w) on (32) which ensure convergent complex eigenvalues are within the segment defined between the intersections of (32) with the parabola in Figure 1 (for the equation of the parabola, refer to [4, 5]). The *uppermost* point (A'_2) defining the segment is given by (34) and (35), whereas the *lowermost* point (A'_1) is given by (36).

$$A'_2 = \left(\phi_{\max}^{(comp)}, w_{\max}^{(comp)}\right) = \begin{cases} \left(\left[\dfrac{\phi_0}{w_0}\right], [1]\right) & \text{if } \dfrac{w_0}{\phi_0} > \dfrac{1}{4} \\ \left(\phi^{(par2)}, w^{(par2)}\right) & \text{otherwise} \end{cases} \tag{34}$$

$$\left(\phi^{(par2)}, w^{(par2)}\right) = \left(\left[\frac{\phi_0}{(\phi_0 - w_0)^2}\cdot\left((\phi_0 + w_0) + 2\cdot\sqrt{\phi_0\cdot w_0}\right)\right], \left[\frac{w_0}{(\phi_0 - w_0)^2}\cdot\left((\phi_0 + w_0) + 2\cdot\sqrt{\phi_0\cdot w_0}\right)\right]\right) \tag{35}$$

$$\boxed{\begin{array}{l} A'_1 = \left(\phi_{\min}^{(comp)}, w_{\min}^{(comp)}\right) = \left(\phi^{(par1)}, w^{(par1)}\right) \\ \left(\phi^{(par1)}, w^{(par1)}\right) = \left(\left[\frac{\phi_0}{(\phi_0 - w_0)^2} \cdot \left((\phi_0 + w_0) - 2 \cdot \sqrt{\phi_0 \cdot w_0}\right)\right], \left[\frac{\phi_0}{(\phi_0 - w_0)^2} \cdot \left((\phi_0 + w_0) - 2 \cdot \sqrt{\phi_0 \cdot w_0}\right)\right]\right) \end{array}} \tag{36}$$

Let us call the original settings A_0, as in (37):

$$\boxed{A_0 = (\phi_0, w_0)} \tag{37}$$

To ensure convergence by constricting a *CPSO*, setting a constriction factor that complies with (38) suffices:

$$\boxed{0 < \chi < \left(\frac{\|A_2\|}{\|A_0\|} = \frac{\phi_{\max}^{(conv)}}{\phi_0} = \frac{w_{\max}^{(conv)}}{w_0}\right)} \tag{38}$$

If the user wants to ensure that the constricted coefficients fall within the convergent complex region, setting a constriction factor that complies with (39) suffices:

$$\boxed{\left(\frac{\|A'_1\|}{\|A_0\|} = \frac{\phi_{\min}^{(comp)}}{\phi_0} = \frac{w_{\min}^{(comp)}}{w_0}\right) < \chi < \left(\frac{\|A'_2\|}{\|A_0\|} = \frac{\phi_{\max}^{(comp)}}{\phi_0} = \frac{w_{\max}^{(comp)}}{w_0}\right)} \tag{39}$$

Refer to Figure 3 for a visualization of the locations of points A_1, A_2, A'_1 and A'_2 in the 'ϕ–w' plane for three different initially set coefficients $A_0 = (\phi_0,w_0)$ in *CCPSO*.

3.2.1 Ensuring convergence (summary)

A procedure that ensures convergence can be reduced to (40) and (41):

$$\boxed{w_{\max}^{(conv)} = \begin{cases} 1 & \text{if } \frac{w_0}{\phi_0} > \frac{1}{4} \\ \frac{2 \cdot w_0}{\phi_0 - 2 \cdot w_0} & \text{otherwise} \end{cases}} \tag{40}$$

$$\boxed{0 < \chi < \frac{w_{\max}^{(conv)}}{w_0}} \tag{41}$$

Note that $\chi_{\max}$ is greater than 1 if the originally set (ϕ_0,w_0) pair is inside the convergence region. For χ approaching the upper limit, convergence speed decreases, but the opposite is not true for χ approaching 0.

3.2.2 Ensuring complex eigenvalues (summary)

A procedure that ensures complex eigenvalues for the surrogate system can be reduced to (42) to (44):

$$\boxed{w_{\max}^{(comp)} = \begin{cases} 1 & \text{if } \frac{w_0}{\phi_0} > \frac{1}{4} \\ \left[\frac{w_0}{(\phi_0 - w_0)^2}\right] \cdot \left[(\phi_0 + w_0) + 2 \cdot \sqrt{\phi_0 \cdot w_0}\right] & \text{otherwise} \end{cases}} \tag{42}$$

$$\boxed{w_{\min}^{(comp)} = \left[\frac{w_0}{(\phi_0 - w_0)^2}\right] \cdot \left[(\phi_0 + w_0) - 2 \cdot \sqrt{\phi_0 \cdot w_0}\right]} \tag{43}$$

$$\boxed{\frac{w_{\min}^{(comp)}}{w_0} < \chi < \frac{w_{\max}^{(comp)}}{w_0}} \tag{44}$$

Note that $\chi_{\max}$ is greater than 1 if the originally set (ϕ_0,w_0) pair is already inside the complex region. Here convergence speed indeed increases for decreasing values of χ within the whole range defined in (44). As opposed to the original *Type 1" COPSO* in [6], the calculation of the range of ϕ which results in complex eigenvalues of the surrogate system is simple and covers the whole complex region. In addition, an interval is defined for χ from which any value selected leads to convergence, and the lower the value the faster the convergence. Hence the meaning of choosing values closer to the limits of the interval is obvious.

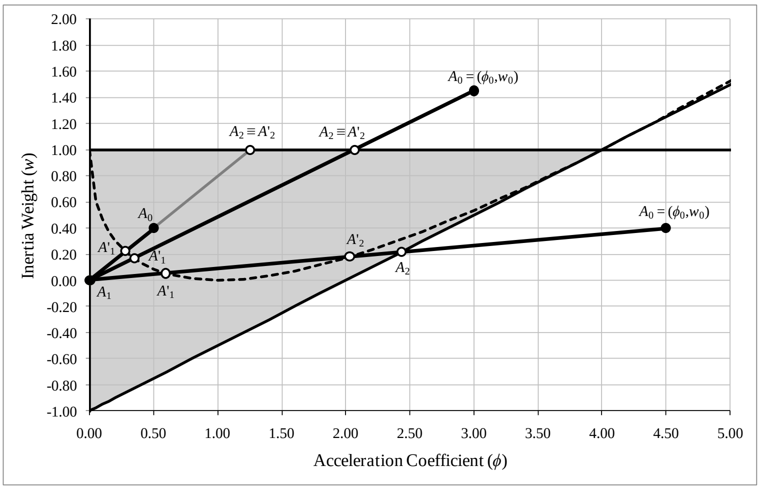

Figure 3: Three different initially set coefficients (ϕ_0,w_0) and the corresponding points A_1, A_2, A'_1 and A'_2 along their corresponding scaling line in *CCPSO*.

4 Concluding remarks

The *Type 1* and *Type 1" COPSO* were analyzed in some detail to clarify the underlying concepts. The *COPSO* was proposed by Clerc and Kennedy [6] as a means to ensure convergence given an *initially set acceleration coefficient* (ϕ_0) and no inertia weight. While *Type 1* constriction applies to any $\phi_0 > 0$, *Type 1"* is said to be valid only for $\phi_0 \in (\phi_{min}, \phi_{max})$ (as it turns out, it is also valid for any $\phi_0 > 0$). Common mistakes when simultaneously applying constriction factor (χ) and inertia weight (w) in *CCPSO* were pointed out and discussed. The convergence studies carried out for the *CPSO* in [5] enabled us to provide equations for the calculation of *Type 1* and *Type 1"* constriction factors when there is also inertia weight $w \neq 1$ in the system. In addition, the equations provided allow for any initially set ϕ_0 and for better control over the convergence speed: if complex convergent eigenvalues are ensured, an interval for χ is calculated where the closer to the lower limit the faster the convergence and the closer to the upper limit the slower the convergence. Since χ can be greater than one, the proposed constriction factor is in fact a scaling factor. Thus, convergence may be delayed yet still ensured if the initially set (ϕ_0,w_0) pair results in too fast a convergence.

Note that this work is limited to the deterministic particle, and the proposed equations are obtained using equations and results from [5]. The extension to the full PSO system is simple: replacing ϕ by ϕ_{max} ensures convergence. The study of the effects of allowing parts of the ϕ range to leave the convergence region is beyond the scope of this paper.

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